Minimal uncertainty states for the rotation and allied groups

1977 J. Phys. A: Math. Gen. 10 1837

(http://iopscience.iop.org/0305-4470/10/11/012)

J. Phys. A: Math. Gen., Vol. 10, No. 11, 1977. Printed in Great Britain. © 1977

Minimal uncertainty states for the rotation and allied groups

R Delbourgo

Department of Physics, University of Tasmania, Hobart, Tasmania, Australia

Received 2 June 1977, in final form 5 July 1977

Abstract. We show that the measure of angular momentum uncertainty is the invariant $(\Delta J)^2 \equiv \langle \boldsymbol{J}^2 \rangle - \langle \boldsymbol{J} \rangle^2$, rather than sums of products like $\Delta J_1 \Delta J_2$, and that the critical states which minimise ΔJ are eigenstates of maximum weight of $\boldsymbol{J}.\boldsymbol{n}$. We also determine the critical states for the associated groups E(2), O(2, 1), O(4) and E(3). Finally we construct coherent O(3) spin states $|z\rangle$ which are superpositions of normal spin eigenvectors and which tend to the classical limit as $|z| \to \infty$.

1. Introduction

Every basic treatise on quantum mechanics features Heisenberg's position–momentum indeterminacy principle and its deep implications for the physical measurement process. Usually, the example of the simple harmonic oscillator is quoted for the reason that the ground level is the select stationary state which minimises the uncertainty product $\Delta x \Delta p$ at $\frac{1}{2}\hbar$, in contrast to the excited states. More modern books go on to discuss coherent superpositions of energy eigenstates which retain the minimal uncertainty (Glauber 1963), and indeed an extensive literature (see Jackiw 1968, and Mathews and Eswaran 1974 for further references) has arisen over the number-phase uncertainty product appropriate to these coherent states.

So far as angular momentum $\boldsymbol{J}$ is concerned, most elementary books limit themselves to observing that only a single component $\boldsymbol{J}.\boldsymbol{n}$ can be measured at once, with the resulting indeterminacy in the other two perpendicular components described by a vector model in which $\boldsymbol{J}$ rotates about the direction $\boldsymbol{n}$ of quantisation. In this paper we wish to study the indeterminacy question for $\boldsymbol{J}$ in greater depth and try to decide what is the appropriate measure for it; certainly singling out a particular product like $\Delta J_1\, \Delta J_2$ is not enough because of the obvious bias in direction. The problem of finding the relevant uncertainty measure is not confined to O(3), of course, but applies to any group. In § 2 we discuss what is meant in general by 'quasi-classical' states with the 'least indeterminacy' before concentrating on the rotation group in § 3. Our proposal for $(\Delta J)^2 = \langle \boldsymbol{J}^2 \rangle - \langle \boldsymbol{J} \rangle^2$ as probably the correct measure of uncertainty has the virtue that ΔJ is pure scalar, and we determine the critical states which have fixed $\boldsymbol{J}^2$ and least ΔJ. (In § 6 we remove the restriction of constant j and define coherent spin states as appropriate superpositions of j levels.) Using our knowledge of O(3), quasi-classical states for O(2, 1) and E(2) are found in § 4, by continuation and contraction; and in § 5 we do the same for the associated groups O(4) and E(3).

2. Quasi-classical states

The impossibility of precisely predicting in advance the outcome of a single experiment which modifies the initial state of the system finds its natural expression in the quantum mechanical formulation of the uncertainty relations between measurements of incompatible observables. The Heisenberg uncertainty principle, $\Delta x\,\Delta p \geqslant \frac{1}{2}\hbar$, encapsulates the limitations one must face for coordinate X and momentum P measurements and sets a lower limit of precision on the accuracy of experimental determinations of these particular dynamical variables. In this case, the best one can do is to minimise the uncertainty product at $\frac{1}{2}\hbar$, because this is the closest approach to the classical situation of absolute precision; as shown in elementary textbooks this limit is reached for 'minimal states' which satisfy the equation

$$[\Delta p(X-\langle X\rangle) \pm \mathrm{i}\,\Delta x(P-\langle P\rangle)]|\psi\rangle = 0$$

the solutions of which are the well known oscillating Gaussian wave packets—these include the now familiar 'coherent states' of the oscillator (mass m, frequency ω) wherein $\Delta p = m\omega\,\Delta x$ as well.

We would like to pose a similar question for more general commutation relations, appropriate to any Lie algebra, $[F_r, F_s] = \mathrm{i}c_{rst}F_t$ where F are the generators and c are the (real) structure constants. Namely, what are the 'quasi-classical states' for which the uncertainties are minimised? Since the general problem leads us to commutators which are not c numbers, whose expectation values therefore depend on the (normalised) state $|\psi\rangle$ in question, minimisation of $\Delta F_r\,\Delta F_s$ is not guaranteed to equal $\frac{1}{2}|c_{rst}\langle F_t\rangle|$ unless the system happens to be in an eigenstate of the commutator: in that circumstance Heisenberg's derivation of the principle is correct and gives the 'minimal equation'

$$[\Delta F_s(F_r-\langle F_r\rangle) \pm \mathrm{i}\,\Delta F_r(F_s-\langle F_s\rangle)]|\psi\rangle = 0. \tag{I}$$

Otherwise, we are obliged to resort to Jackiw's analytic method (1968) which gives instead the 'critical equation'

$$\left[\left(\frac{F_r-\langle F_r\rangle}{\Delta F_r}\right)^2 + \left(\frac{F_s-\langle F_s\rangle}{\Delta F_s}\right)^2 - 2\right]|\psi\rangle = 0 \tag{II}$$

and to look for normalisable solutions of (II) which minimise $\Delta F_r\,\Delta F_s$. On the other hand, it should be pointed out that if one chooses to minimise the uncertainty product *ratio* $\Delta F_r\,\Delta F_s/|c_{rst}\langle F_t\rangle|$ then Heisenberg's direct method (I) does, in fact, apply.

This preamble leads us to the question of what exactly we have to minimise when we are dealing with several Lie algebra generators, and how we are to sharpen the definition of a quasi-classical state. Two problems pose themselves: (i) should one minimise just the products $\Delta F_r\,\Delta F_s$ or perhaps $\Delta F_r\,\Delta F_s/|c_{rst}\langle F_t\rangle|$ or linear combinations thereof, or even multiple products $\Delta F_r\,\Delta F_s\,\Delta F_t \ldots$? (ii) Having decided on the choice (i), how many operators should one take into account? Or, in other words, how large the Lie algebra? The rational answer to the second question is that one must include all the observables of physics (which comprise a maximally commuting set for resolving any degeneracy) and perform the chosen minimisation on the smallest Lie algebra comprising these observables. In this way we avoid considering operators which are not measurable, whose expectation values have no direct physical content. In this paper we shall have little to say about the grand case (ii), but will concentrate on the choice (i) to be made when faced with a particular Lie algebra. More specifically, we will focus our attention on the quasi-classical states for the rotation group O(3) and allied groups. Having sorted these out, some extensions to more involved cases suggest themselves.

3. A case study: O(3)

Let us record a few simple facts about angular momentum, which, if not always described in the standard texts, are trivially deduced. Let $|jm\rangle_{\boldsymbol{n}}$ be a normalised eigenstate of the angular momentum (in units of $\hbar$) directed along unit vector $\boldsymbol{n}$,

$$\boldsymbol{J}\cdot\boldsymbol{n}|jm\rangle_{\boldsymbol{n}} = m|jm\rangle_{\boldsymbol{n}}, \qquad \boldsymbol{J}^2|jm\rangle_{\boldsymbol{n}} = j(j+1)|jm\rangle_{\boldsymbol{n}}. \tag{1}$$

Then, for such a state,

$$\langle \boldsymbol{J}\rangle = m\boldsymbol{n}, \qquad (\Delta J_1)^2 = \tfrac{1}{2}(1-n_1^2)[j(j+1)-m^2],$$

etc, and

$$(\Delta \boldsymbol{J})^2 = \langle \boldsymbol{J}^2\rangle - \langle \boldsymbol{J}\rangle^2 = \langle \boldsymbol{J}^2\rangle - \langle \boldsymbol{J}\cdot\boldsymbol{n}\rangle^2 = j(j+1)-m^2. \tag{2}$$

The case $n_3=1$, $n_1=n_2=0$ is the most familiar, when

$$\begin{aligned} &\langle J_1\rangle = \langle J_2\rangle = 0, \qquad \langle J_3\rangle = m, \\ &\Delta J_1 = \Delta J_2 = [\tfrac{1}{2}j(j+1) - \tfrac{1}{2}m^2]^{1/2}, \qquad \Delta J_3 = 0 \end{aligned} \tag{3}$$

and the product $\Delta J_1\,\Delta J_2$ is minimised at $\frac{1}{2}j$ by taking $|m|=j$; the other uncertainty products $\Delta J_2\,\Delta J_3$ and $\Delta J_3\,\Delta J_1$ of course being zero. This illustrates the fact that for a compact Lie group like O(3) where the ΔF are bounded it is quite easy to arrange for a number of indeterminacy products to vanish identically by simply diagonalising as many generators as simultaneously possible (the Cartan sub-algebra defining the rank).

If we plot the $\Delta\boldsymbol{J}$ in three dimensions we recognise the products $\Delta J_1 \Delta J_2$, etc, as uncertainty areas. Several natural measures of total uncertainty come to mind, namely

the 'uncertainty volume', $\Delta^2 J \equiv \Delta J_1\,\Delta J_2\,\Delta J_3$ (4*a*)

the 'uncertainty surface', $\Delta^2 J \equiv [(\Delta J_1\,\Delta J_2)^2 + (\Delta J_2\,\Delta J_3)^2 + (\Delta J_3\,\Delta J_1)^2]^{1/2}$ (4*b*)

the 'uncertainty radius', $\Delta J \equiv [(\Delta J_1)^2 + (\Delta J_2)^2 + (\Delta J_3)^2]^{1/2}$. (4*c*)

We note that† only the latter is a true scalar. $\Delta^3 J$ becomes a rotational invariant in the infinitesimal limit only, while the components $\Delta J_i\,\Delta J_j$ in $\Delta^2 J$ can be regarded as defining a surface vector. Since it is also true that

$$\Delta J_1\,\Delta J_2 \geqslant \tfrac{1}{2}|\langle J_3\rangle| \tag{5}$$

etc, more exotic measures of uncertainties can be contemplated, involving weightings by expectation values, for instance

$$\delta^3 J \equiv (\Delta J_1\,\Delta J_2\,\Delta J_3)^2/|\langle J_1\rangle\langle J_2\rangle\langle J_3\rangle| \tag{6a}$$

$$\delta^2 J \equiv [(\Delta J_1\,\Delta J_2)^2/|\langle J_3\rangle|^2 + \text{cyclic}]^{1/2}, \tag{6b}$$

† From the inequalities between geometric and arithmetic means, note also that $(\Delta^2 J) \geqslant 3^{1/2}(\Delta^3 J)^{2/3}$ and $(\Delta J)^3 \geqslant 3^{3/2}\Delta^3 J$.

and so on. Because we can analyse the case of angular momentum in some detail, fortunately we are able to decide which uncertainty measure is the most appropriate and obtain some clues as to how to proceed with more complicated groups.

It is immediately apparent that minimising the uncertainty volume gives precious little information; thus $\Delta^3 J \equiv 0$ when we are dealing with eigenstates of *any* individual component J_i. Unfortunately too, as a measure of indeterminacy it depends on one's choice of axes. For instance, if we select $\boldsymbol{n} = (1, 1, 1)/\sqrt{3}$ and $m = j$, then according to (2),

$$\Delta J_1 = \Delta J_2 = \Delta J_3 = (j/3)^{1/2}$$

giving $\Delta^3 J = (j/3)^{3/2}$ for this choice of quantisation direction. On the other hand, referred to axes parallel and perpendicular to $\boldsymbol{n}$, $(\Delta^3 J)_{\boldsymbol{n}}$ is zero. For these reasons we shall reject measure (4*a*) as totally unsatisfactory.

Turning to (4*b*), we know from the usual direct derivation of the relations (5) that

$$\Delta^2 J \geqslant \tfrac{1}{2}[\langle J_3\rangle^2 + \langle J_2\rangle^2 + \langle J_1\rangle^2]^{1/2} = \tfrac{1}{2}[j(j+1) - (\Delta J)^2]^{1/2}. \tag{7}$$

In fact, we can remove the equality sign from (7) since the lower bound can only be attained for states which satisfy

$$c_1(J_1 - \langle J_1\rangle)|\psi\rangle = c_2(J_2 - \langle J_2\rangle)|\psi\rangle = c_3(J_3 - \langle J_3\rangle)|\psi\rangle$$

where c_2/c_1, c_3/c_2 and c_1/c_3 are all imaginary, which is clearly impossible. (The case when one of the uncertainties, say ΔJ_3, is zero, causing some c_i to vanish, has to be examined separately; here the answer is already known, namely $\Delta^2 J$ is indeed minimised at $\frac{1}{2}j$ for $|\psi\rangle = |jj\rangle$. Then the equality sign in (7) does indeed apply.) Because there is a strict inequality in (7) for all $\Delta J_i \neq 0$, this implies a failure of the direct method, and to minimise $\Delta^2 J$ we must resort to Jackiw's analytic method which supplies a weaker condition on the wavefunction. Here one finds the critical equation

$$\{(J_1 - \langle J_1\rangle)^2[(\Delta J_2)^2 + (\Delta J_3)^2] + \text{cyclic terms} - 2(\Delta^2 J)^2\}|\psi\rangle = 0 \tag{8}$$

which can be cast in the form

$$[\nu_1^2(J_1^2 - \langle J_1\rangle^2) + \text{cyclic} - (\Delta^2 J)^2(\Delta J)^{-2}]|\psi\rangle = 0 \tag{9}$$

where the components of the unit vector $\boldsymbol{\nu}$ are given by

$$\nu_1^2 = [(\Delta J_2)^2 + (\Delta J_3)^2]/2(\Delta J)^2 \qquad \text{etc.} \tag{10}$$

For reasons of symmetry $\Delta^2 J$ attains its physical extremum† when $\boldsymbol{\nu} = (1, 1, 1)/\sqrt{3}$ with all ΔJ_i equal. As shown in the appendix the solution of (9) with the least $\Delta^2 J$ is none other than $|jj\rangle_{\boldsymbol{\nu}}$ and yields $\Delta^2 J = j$. On the other hand, we clearly do better for the uncertainty surface by taking one uncertainty to vanish (say ΔJ_3) since then $\Delta^2 J$ is as small as $\frac{1}{2}j$. As it is unreasonable to say that an eigenstate of J_3 or J_1 or J_2 has a smaller uncertainty than an eigenstate of $\boldsymbol{J}.\boldsymbol{n}$ we conclude that $\Delta^2 J$ is not a good measure of indeterminacy either, primarily because of its lack of rotational invariance.

The last simple case to examine is the uncertainty radius (4*c*). Here the critical states are given by

$$(\boldsymbol{J} - \langle \boldsymbol{J}\rangle)^2|\psi\rangle = (\Delta J)^2|\psi\rangle. \tag{11}$$

† The hypothetical extremum $\Delta^2 J = 0$ cannot occur physically unless $j = 0$. Other possibilities like $\boldsymbol{\nu} = (1, 1, 0)/\sqrt{2}$ can be ruled out as they imply that $\Delta J_1 = \Delta J_2 = 0$, an impossibility. The case $\Delta J_3 = 0$, $\Delta J_1 = \Delta J_2 \neq 0$ with $\boldsymbol{\nu} = (1, 1, \sqrt{2})/2$ has already been considered.

In the appendix we prove that the solution of (11) is the naive answer $|\psi\rangle = |jj\rangle_n$ which yields the minimal $\Delta J = j^{1/2}$. This answer has the virtue that it does not depend on the orientation of axes, and we therefore suggest that the radius ΔJ is much the best measure of uncertainty. It can be argued that what we are proposing for angular momentum contradicts the standard procedure for position–momentum uncertainty. However, it must be pointed out that the Weyl group of Heisenberg commutators is different from O(3) and is, moreover, invariant under the scaling $X \to \lambda X$, $P \to P/\lambda$—hence the area $\Delta x \Delta p$ is the natural measure of uncertainty, not $(\Delta x)^2 + (\Delta p)^2$. For $\boldsymbol{J}$ there is no such scaling invariance, rather the relative weights of $\boldsymbol{J}$ components are absolutely fixed via the Casimir operator $\boldsymbol{J}^2$. And that is why we are advancing ΔJ as the relevant measure, rather than $\Delta^2 J$ or $\Delta^3 J$.

The more sophisticated measures of uncertainty (6*a*) and (6*b*) would require an analysis in their own right were it not for their lack of rotational symmetry. Without examining them in detail we note that for $|jm\rangle_n$ states considered in (1) and (2),

$$\delta^3 J = [\tfrac{1}{2}j(j+1) - \tfrac{1}{2}m^2]^3[(1-n_1^2)(1-n_2^2)(1-n_3^2)/n_1 n_2 n_3 |m|^3]$$

$$\delta^2 J = [\tfrac{1}{2}j(j+1) - \tfrac{1}{2}m^2][(1-n_1^2)(1-n_2^2)/n_3^2 + \text{cyclic}]^{1/2}$$

become infinite when quantisation along conventional axes (x or y or z) is made, and they achieve their minimal values of $1/3\surd 3$ and $j\surd 3$ respectively along axis $\boldsymbol{n} = (\pm 1, \pm 1, \pm 1)/\surd 3$ when $m = j$. This suffices to demonstrate their uselessness although one could no doubt arrive at the same conclusion by investigating the critical states of $\delta^3 J$ and $\delta^2 J$.

In a nutshell, the final result of this section is that the states of minimal uncertainty are $|jj\rangle_n$ and possess the uncertainty radius $\Delta J = j^{1/2}$. Of course there is no uncertainty in $\boldsymbol{J}^2$.

4. E(2) and O(2, 1)

We can approach the Euclidean group by a contraction of O(3); by putting $P_1 = cJ_1$, $P_2 = cJ_2$ and taking the limit $c \to 0$ in the resulting commutators

$$[P_1, P_2] = \mathrm{i}c^2 J_3, \qquad [P_2, J_3] = \mathrm{i}P_1, \qquad [J_3, P_1] = \mathrm{i}P_2 \tag{12}$$

one arrives at the E(2) Lie algebra. The Casimir

$$\boldsymbol{J}^2 = (P_1^2 + P_2^2)/c^2 + J_3^2 \to \infty$$

is this limit, so by fixing $p = jc$ and finite, p^2 remains as the eigenvalue of the E(2) Casimir $\vec{P}^2$. The relevant uncertainty here is ΔP and since, for the critical angular momentum states $|jj\rangle_n$ already found,

$$(\Delta P_1)^2 + (\Delta P_2)^2 = c^2[(\Delta \boldsymbol{J})^2 - (\Delta J_3)^2] = \tfrac{1}{2}c^2 j(1+n_3^2) = \tfrac{1}{2}pc(1+n_3^2),$$

we see that as $c \to 0$, $\Delta p \to 0$ for all $\boldsymbol{n}$. However, $(\Delta J_3)^2 = \tfrac{1}{2}(1-n_3^2)p/c \to \infty$ unless $n_3 = 1$. These minimal states of E(2) are none other than the eigenstates $|\psi\rangle$ of the translation group, but because they are not strictly normalisable we must analyse them more carefully.

Form the normalised wave packet

$$|f_{\vec{k}}\rangle = \int \mathrm{d}\vec{k}\, f(\vec{k})|\vec{k}\rangle, \qquad \int |f(\vec{k})|^2\, \mathrm{d}\vec{k} = 1.$$

Therefore

$$(\Delta\vec{P})^2 = \int \vec{k}^2|f(\vec{k})|^2\,d\vec{k} - \left(\int \vec{k}|f(\vec{k})|^2\,d\vec{k}\right)^2.$$

In the limit that f becomes a distribution: $|f|^2 \to \delta(\vec{p}-\vec{k})$, we strictly get $\Delta p \to 0$, but the problematic quantity is now $\langle J_3\rangle = -\frac{1}{2}\int f^*(\mathrm{i}\partial f/\partial\phi)\,dk^2\,d\phi$. In order to ensure Hermiticity of J_3, we ought more properly to form periodic wave packets, for example

$$f(\phi) \propto [\exp(-\tfrac{1}{2}\cos^2\tfrac{1}{2}\phi/\sigma)](\sin\tfrac{1}{2}\phi)^{1/2}, \qquad 0<\phi<2\pi$$

with

$$\left.\begin{array}{l}\langle\cos\tfrac{1}{2}\phi\rangle = 0,\ \langle\cos^2\tfrac{1}{2}\phi\rangle \simeq \sigma/2 \\ \langle\cos\phi\rangle \simeq -1+\sigma,\ \langle\sin\phi\rangle = 0\end{array}\right\}\text{as } \sigma\to 0$$

in which case $\Delta J_3 = 0$ trivially. Minimal wave packets of this type correspond to the contracted angular momentum states $|jj\rangle_z$ quantised along the third direction. The other familiar E(2) eigenstates,

$$J_3|p^2m\rangle = m|p^2m\rangle, \qquad (P_1\pm\mathrm{i}P_2)|p^2m\rangle = (\surd p^2)|p^2m\pm1\rangle$$

have the variances $\Delta p = p$ and $\Delta J_3 = 0$ and are not minimal states unless $p=0$. They correspond to taking m finite in $|jm\rangle_z$ while letting $j = p/c \to \infty$.

Turning next to the continuation from O(3) to (2, 1), we put $J_1 = \mathrm{i}K_1$, $J_2 = \mathrm{i}K_2$ to arrive at

$$[K_1, K_2] = -\mathrm{i}J_3, \qquad [J_3, K_1] = \mathrm{i}K_2, \qquad [J_3, K_2] = -\mathrm{i}K_1 \tag{13}$$

wherein $\vec{K}$ and J_3 are now the Hermitian generators. The O(2, 1) unitary representations are characterised in the same way as angular momentum

$$\begin{aligned}(J_3^2-K_1^2-K_2^2)|jm\rangle &= j(j+1)|jm\rangle \\ J_3|jm\rangle &= m|jm\rangle\end{aligned} \tag{14}$$

but the range of j and m values is different. There are:

(*a*) the discrete series $D^{\pm}$ where

$$m = \pm(j+1), \pm(j+2), \ldots \qquad j = -\tfrac{1}{2}, 0, \tfrac{1}{2}, \ldots; \tag{15a}$$

(*b*) the principal series D^{p} where

$$m = \text{integer or half-integer} \qquad j+\tfrac{1}{2} = \text{pure imaginary} = \mathrm{i}\rho; \tag{15b}$$

(*c*) the complementary series D^{c} where

$$m = \text{integer} \quad \text{and} \quad -1<j<0. \tag{15c}$$

Continuous cases (*b*) and (*c*) have a negative Casimir and the case $j=-1$ gives a trivial representation. Having learnt the lesson of O(3), we suggest that the quasi-classical states here are the ones which minimise

$$\Delta K \equiv [(\Delta K_1)^2 + (\Delta K_2)^2 - (\Delta J_3)^2]^{1/2}. \tag{16}$$

If we stick to the standard states (14) for the moment, this means that, since $\Delta K = [m^2 - j(j+1)]^{1/2}$, for (*a*) $D^{\pm}$, ΔK minimises at $(j+1)^{1/2}$ when $m = \pm(j+1)$; while for (*b*) and (*c*) $D^{\mathrm{p,c}}$, ΔK minimises at $(\frac{1}{4}+\rho^2)^{1/2}$ or $[-j(j+1)]^{1/2}$ when $m=0$. The O(2, 1) invariance of ΔK ensures that the same conclusions are true for any other direction of quantisation determined by a 'unit vector' $\boldsymbol{n} = (n_1, n_2, n_3)$; $n_3^2 - n_1^2 - n_2^2 = 1$. The formulae

$$(\Delta K_{1,2})^2 = \tfrac{1}{2}(1 + n_{1,2}^2)[m^2 - j(j+1)]$$
$$(\Delta J_3)^2 = \tfrac{1}{2}(n_3^2 - 1)[m^2 - j(j+1)] \tag{17}$$

are the appropriate continuations of (2).

5. O(4) and E(3)

O(4) being locally isomorphic to O(3) × O(3), we can immediately apply the results of § 3 to it. In the usual way, form two vectors $\boldsymbol{J} = (J_{23}, J_{31}, J_{12})$, $\boldsymbol{K} = (J_{14}, J_{24}, J_{34})$ out of the O(4) generators $J_{\mu\nu}$, and in the ensuing commutators,

$$\boldsymbol{J} \times \boldsymbol{J} = \boldsymbol{K} \times \boldsymbol{K} = \mathrm{i}\boldsymbol{J}, \qquad \boldsymbol{J} \times \boldsymbol{K} = \mathrm{i}\boldsymbol{K} \tag{18}$$

take independent linear combinations $\boldsymbol{J}(\pm) = \tfrac{1}{2}(\boldsymbol{J} \pm \boldsymbol{K})$ to obtain

$$\boldsymbol{J}(\pm) \times \boldsymbol{J}(\pm) = \mathrm{i}\boldsymbol{J}(\pm), \qquad \boldsymbol{J}(\pm) \times \boldsymbol{J}(\mp) = 0. \tag{19}$$

The O(4) states are, in effect, labelled by the $\mathrm{O}_\pm(3)$ Casimirs $j_\pm(j_\pm + 1)$ and written $|j_+m_+, j_-m_-\rangle_{\boldsymbol{n}(+)\boldsymbol{n}(-)}$ when quantised along axes $\boldsymbol{n}(+)$ and $\boldsymbol{n}(-)$. The O(4) Casimirs themselves are given by

$$\Sigma^2 = \tfrac{1}{4}J_{\mu\nu}J_{\mu\nu} = \tfrac{1}{2}(\boldsymbol{J}^2 + \boldsymbol{K}^2) = \boldsymbol{J}(+)^2 + \boldsymbol{J}(-)^2 = j_+(j_+ + 1) + j_-(j_- + 1)$$
$$\Pi^2 = |\tfrac{1}{8}\epsilon_{\mu\nu\kappa\lambda}J_{\mu\nu}J_{\kappa\lambda}| = |\boldsymbol{J}\cdot\boldsymbol{K}| = |\boldsymbol{J}(+)^2 - \boldsymbol{J}(-)^2| = |j_+(j_+ + 1) - j_-(j_- + 1)|. \tag{20}$$

These are the invariant dispersions we are suggesting should be minimised in trying to define a quasi-classical state. The smallest dispersions occur for $m_\pm = j_\pm$ whereupon

$$(\Delta\Sigma)^2 \equiv \tfrac{1}{2}[(\Delta\boldsymbol{J})^2 + (\Delta\boldsymbol{K})^2] = \langle\tfrac{1}{4}J_{\mu\nu}J_{\mu\nu}\rangle - \tfrac{1}{4}\langle J_{\mu\nu}\rangle\langle J_{\mu\nu}\rangle$$
$$= j_+ + j_- = -1 + (\tfrac{1}{2} + \Sigma^2 + \Pi^2)^{1/2} + (\tfrac{1}{2} + \Sigma^2 - \Pi^2)^{1/2} \tag{21}$$
$$(\Delta\Pi)^2 \equiv |\langle\boldsymbol{J}\cdot\boldsymbol{K}\rangle - \langle\boldsymbol{J}\rangle\cdot\langle\boldsymbol{K}\rangle| = \tfrac{1}{8}\epsilon_{\mu\nu\kappa\lambda}|\langle J_{\mu\nu}J_{\kappa\lambda}\rangle - \langle J_{\mu\nu}\rangle\langle J_{\kappa\lambda}\rangle|$$
$$= |j_+ - j_-| = |(\tfrac{1}{2} + \Sigma^2 + \Pi^2)^{1/2} - (\tfrac{1}{2} + \Sigma^2 - \Pi^2)^{1/2}|. \tag{22}$$

In fact, these critical states are eigenfunctions of $\tfrac{1}{2}J_{\mu\nu}n_{\mu\nu}$ and of $\tfrac{1}{4}\epsilon_{\mu\nu\kappa\lambda}J_{\mu\nu}n_{\kappa\lambda}$ where the 'unit tensor' $n_{\mu\nu}$ is composed of $\boldsymbol{n}_+$ and $\boldsymbol{n}_-$ similarly to the way $J_{\mu\nu}$ is broken up into $\boldsymbol{J}(+)$ and $\boldsymbol{J}(-)$. If we do not place any further restrictions on Π and Σ that is all there is to say. However, if we require further a vanishing pseudo-scalar Casimir Π, then $j_+ = j_-(= j$ say) and $\Delta\Sigma = (2j)^{1/2}$ is the minimal scalar uncertainty.

The contraction to E(3) is fairly straightforward. Put $\boldsymbol{P} = c\boldsymbol{K}$ as in § 4 and let $c \to 0$ with $\boldsymbol{J}$ held finite, but $|\boldsymbol{K}| \to \infty$ somehow. Because $\boldsymbol{J} = \boldsymbol{J}(+) + \boldsymbol{J}(-)$, this means we must let j_+ and $j_- \to \infty$ with $j_+ - j_- = \lambda$ held fixed. In this limit, the E(3) Casimirs $\boldsymbol{P}^2$ and $\boldsymbol{J}\cdot\boldsymbol{P}$ arise from O(4) Casimirs $\Sigma^2 \to \tfrac{1}{2}\boldsymbol{P}^2/c^2$, $\Pi^2 \to \boldsymbol{J}\cdot\boldsymbol{P}/c$, and thus, putting $p = 2cj_+$,

$$\boldsymbol{P}^2 \to 2c^2[j_+(j_+ + 1) + j_-(j_- + 1)] \to p^2$$
$$\boldsymbol{P}\cdot\boldsymbol{J} \to c\Pi^2 \to c[j_+(j_+ + 1) - j_-(j_- + 1)] \to \lambda p. \tag{23}$$

The E(3) variances tend, as $c \to 0$, to

$$\begin{aligned} \Delta p &= c[(\Delta \boldsymbol{K})^2]^{1/2} \to 2c\, \Delta\Sigma = 2c(2j_+)^{1/2} \to 0 \\ p\Delta\lambda &\equiv |\langle \boldsymbol{P} . \boldsymbol{J}\rangle - \langle \boldsymbol{P}\rangle . \langle \boldsymbol{J}\rangle| = c(\Delta\Pi)^2 \to c|\lambda| \to 0 \end{aligned} \tag{24}$$

i.e. the minimal states $|\boldsymbol{p}\lambda\rangle$, eigenstates of $\boldsymbol{P}^2$ and $\boldsymbol{P} . \boldsymbol{J}$, can have vanishing dispersion after strongly localised wave packets are constructed in the manner of § 4. We note that the angular momentum dispersion becomes infinite

$$\begin{aligned} (\Delta J)^2 &= \langle \boldsymbol{J}^2\rangle - \langle \boldsymbol{J}\rangle . \langle \boldsymbol{J}\rangle \\ &= j_+(j_+ + 1) + j_-(j_- + 1) + 2j_+j_-\boldsymbol{n}_+ . \boldsymbol{n}_- - (\boldsymbol{n}_+j_+ + \boldsymbol{n}_-j_-)^2 \\ &= (j_+ + j_-) \to \infty \end{aligned}$$

and this seems to be the price we must pay to be certain of the helicity.

6. Coherent spin states

A great deal is known about coherent states for the harmonic oscillator, but far less about coherent states for angular momentum. (See Bacry *et al* (1976) for a recent review.) In making an analogy between J_3 and the number operator, and between J_- and the annihilation operator, Radcliffe (1971) was able to construct a coherent spin basis and demonstrate its usefulness in the context of statistical mechanics for ferromagnetic systems. (Subsequently Kolodziejczyk and Ryter (1974) showed that a state of minimal uncertainty $(\Delta J_1)(\Delta J_2)$ was only possible for Radcliffe's ground state, i.e. $|J_3| = j$, as we proved more directly in § 3.) His formulation fixed with certainty the Casimirs $j_1, j_2, \ldots$ of a series of angular momenta. Owing to the finite degeneracy of spin vectors having no counterpart in the oscillator we would advocate the alternative course of equating j, *not* J_3, with the number operator of the oscillator and then constructing a coherent basis as an infinite superposition of different spin-j states. We can minimise ΔJ by superposing $|jj\rangle_{\boldsymbol{n}}$ states, so one rather obvious possibility is to form the linear combination

$$|\alpha\rangle = \sum_j \alpha^j [\exp(-\tfrac{1}{2}|\alpha|^2)]|jj\rangle/(j!)^{1/2} \tag{25}$$

for Bose systems say, from which it is readily established that

$$\langle \boldsymbol{J}^2\rangle = |\alpha|^2(|\alpha|^2 + 2), \qquad \langle \boldsymbol{J}\rangle^2 = |\alpha|^2(|\alpha|^2 + 1), \qquad \Delta J = |\alpha|. \tag{26}$$

A more complete procedure, which includes integer and half-integer spins, is to construct creators $a^\dagger$ and annihilators a of spin-$\frac{1}{2}$

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}; \qquad \alpha, \beta = \uparrow, \downarrow \tag{27}$$

in terms of which $\boldsymbol{J} = \frac{1}{2}a^\dagger \boldsymbol{\sigma} a$. The usual angular momentum states are created as follows:

$$|jm\rangle = \frac{(a_\uparrow^\dagger)^{j+m}(a_\downarrow^\dagger)^{j-m}}{[(j+m)!(j-m)!]^{1/2}}|0\rangle. \tag{28}$$

We can build up coherent spin states, labelled by a two spinor $z=(z_\uparrow, z_\downarrow)$ as follows:

$$|z\rangle=\sum_{jm}\frac{(z_\uparrow)^{j+m}(z_\downarrow)^{j-m}}{[(j+m)!(j-m)!]^{1/2}}|jm\rangle\exp(-\tfrac{1}{2}|z_\uparrow|^2-\tfrac{1}{2}|z_\downarrow|^2)=\mathrm{e}^{a^\dagger z}|0\rangle\mathrm{e}^{-\frac{1}{2}z^\dagger z}. \tag{29}$$

The exponential factors $\mathrm{e}^{-|z|^2/2}$ are included to simplify the 'orthogonality' and 'completeness' relations:

$$\begin{aligned}1&=\int|z\rangle\,\mathrm{d}^2z_\uparrow\,\mathrm{d}^2z_\downarrow\langle z|/\pi^2\\ \langle z'|z\rangle&=\mathrm{e}^{z'^\dagger z}\,\mathrm{e}^{-(z^\dagger z+z'^\dagger z')/2}.\end{aligned} \tag{30}$$

These new states incorporate the simple proposal (25) if we set $z_\downarrow=0$ and sum over half-integer as well as integer j. They may also be related to Radcliffe's states,

$$|\mu\rangle=\sum_m\frac{\mu^{j-m}|jm\rangle}{(1+|\mu|^2)^j}\left(\frac{(2j)!}{(j+m)!(j-m)!}\right)^{1/2} \tag{31}$$

if in (29) we put $z_\uparrow=1$, $z_\downarrow=\mu$ and do not sum over j:

$$|z_\uparrow=1, z_\downarrow=\mu\rangle\sim\frac{\mathrm{e}^{-(1+|\mu|^2)/2}}{(1+|\mu|^2)^j[(2j)!]^{1/2}}|\mu\rangle. \tag{32}$$

Because they are eigenstates of the annihilation operator, it is relatively easy to work out the dispersion of $\boldsymbol{J}^2$. One finds

$$\begin{aligned}\langle z|\boldsymbol{J}|z\rangle&=\tfrac{1}{2}z^\dagger\boldsymbol{\sigma}z\\ \langle z|\boldsymbol{J}^2|z\rangle&=\tfrac{1}{4}z^\dagger z(z^\dagger z+3)\end{aligned}$$

therefore

$$(\Delta J)^2/\langle\boldsymbol{J}^2\rangle=(1+\tfrac{1}{3}z^\dagger z)^{-1} \tag{33}$$

so as $|z|\to\infty$ the classical limit is reached. Such states could conceivably be useful in describing highly excited molecular rotation levels. Having said that, it is not immediately clear, however, what is the analogue of the oscillator phase operator ϕ, though it must surely be connected to the a and $a^\dagger$ via the structure

$$\cos\phi\sim a^\dagger+a, \qquad \sin\phi\sim a^\dagger-a.$$

We emphasise that all recent studies of coherent spin states (Belissard and Holtz 1974, Hioe 1974, Peremolov 1972, Onofri 1975) base themselves on Radcliffe's original suggestion and confine themselves to fixed j. Thus they differ in an essential way from our proposal (29).

7. Generalisations

We have lowered our sights to O(3) and closely related groups in this paper to deduce that quasi-classical states are those which minimise the uncertainty invariant ΔJ; the vectors turn out to be eigenstates of highest weight of $\boldsymbol{J}\cdot\boldsymbol{n}$. For other Lie algebras a natural generalisation is to look for minima of the quadratic

$$(\Delta F)^2\equiv c_{rst}c_{rsu}(\langle F_tF_u\rangle-\langle F_t\rangle\langle F_u\rangle)$$

and higher-order Casimirs. We anticipate again that the quasi-classical vectors have highest weight. For Lie algebras which are semi-direct products of an Abelian algebra and a compact algebra, our experience with E(2) and E(3) suggests that the minimal states are eigenstates of the Abelian sub-algebra and any other Casimirs. Finally, if one abandons the restriction that Casimirs themselves be precisely known, the method of § 6 suggests a route by which one can construct appropriate coherent states different from those of Peremolov (1972).

Appendix

We wish to solve the equation

$$(\boldsymbol{J}-\langle\boldsymbol{J}\rangle)^2|\psi\rangle=(\Delta J)^2|\psi\rangle$$

wherein $\Delta J_1=\Delta J_2=\Delta J_3$ and therefore $\Delta^2 J=3^{-1/2}(\Delta J)^2$. We look for solutions where $|\psi\rangle$ is the usual eigenstate of $\boldsymbol{J}^2$. The equation then reduces to

$$2\boldsymbol{J}\cdot\langle\boldsymbol{J}\rangle|\psi\rangle=[j(j+1)+\langle\boldsymbol{J}\rangle^2-(\Delta J)^2]|\psi\rangle.$$

Thus $|\psi\rangle$ is in an eigenstate of $\boldsymbol{J}\cdot\boldsymbol{n}$ where $\boldsymbol{n}$ is a unit vector parallel to $\langle\boldsymbol{J}\rangle$ and therefore

$$\boldsymbol{J}\cdot\langle\boldsymbol{J}\rangle|\psi\rangle=m|\langle\boldsymbol{J}\rangle||\psi\rangle. \tag{A.1}$$

The equality of uncertainties entails that $\langle\boldsymbol{J}\rangle=m(\pm1,\pm1,\pm1)/\sqrt{3}$ and correspondingly $(\Delta J_1)^2=(\Delta J_2)^2=(\Delta J_3)^2=\frac{1}{3}[j(j+1)-m^2]$ giving $\Delta^2 J=[j(j+1)-m^2]$. The critical state with the least $\Delta^2 J$ then corresponds to the selection $m=j$.

If we abandon the constraint $\Delta J_1=\Delta J_2=\Delta J_3$, the solution to (A.1) is simply that $\boldsymbol{J}\cdot\boldsymbol{n}|\psi\rangle=m|\psi\rangle$ with $\langle\boldsymbol{J}\rangle=m\boldsymbol{n}$, namely the case considered at the beginning of § 3.

References

Bacry H, Grossman A and Zak A 1976 *Lecture Notes in Physics* vol. 50 (Berlin: Springer)
Belissard J and Holtz R 1974 *J. Math. Phys.* **15** 1275
Glauber R J 1963 *Phys. Rev.* **131** 2766
Hioe F T 1974 *J. Math. Phys.* **15** 1174
Jackiw R 1968 *J. Math. Phys.* **9** 339
Kolodziejczyk L and Ryter A 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 213
Mathews P M and Eswaran K 1974 *Nuovo Cim.* B **19** 99
Onofri E 1975 *J. Math. Phys.* **16** 1087
Peremolov A M 1972 *Commun. Math. Phys.* **26** 222
Radcliffe J M 1971 *J. Phys. A: Gen Phys.* **4** 313